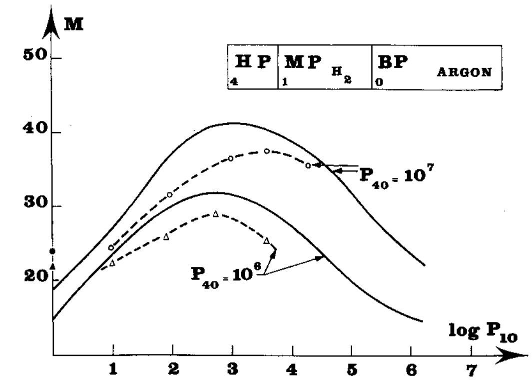

Fig. 2 Mach number vs logarithm of the ratio of intermediate chamber pressure to test chamber pressure for two values of the ratio of driver chamber pressure to test chamber pressure, ——— theoretical curve, ----- experimental curve, ○ and △ experimental points. ● ▲ experimental points without an intermediate chamber.

The experiments here reported deal with argon ($P_0 = 1$ torr) undergoing a shock of $M \simeq 30$. A continuous writing streak camera enables measurement with 1% of the velocity of the wave 7 m away from D_2 diaphragm (Fig. 1a). Two photomultipliers whose distances to D_2 are 9.8 m and 10.2 m, respectively, give the velocity and the test time (Fig. 1b).

The existence of an optimum P_1 pressure has been checked. Figure 2 shows that the optimum value of P_1 is obtained for $P_{10} = 10^3$ when $P_{40} = 10^6$. From experiments at other values of P_{40}, it may be deduced that P_{10} optimum $\simeq P_{40}{}^{1/2}$.

Measurement of the attenuation of the shock wave velocity is performed through a 35 GHz microwave system. The microwave is reflected by the shock front along the axis of the tube. Because of the precursor phenomena which travels ahead of a strong shock wave, this Doppler effect method gives the correct value of the velocity when the microwave frequency is high enough for the microwave to be reflected near the pressure front. Figure 3 gives the attenuation of the velocity as a function of the distance to D_2 diaphragm for different values of P_0.

On Fig. 3 are shown also the results from some high-performance shock tubes. Arc driven shock tubes[2] give very good velocity and test time. Yet we must observe that helium driver gas is very hot ($T \geqq 2 \times 10^4$ °K). In radiative transfer studies driver gas radiation would have to be taken into account. Highest performances reached until now, have been achieved by explosively driven shock tubes.[3] In these tubes, shock heated gas is quickly disturbed by the expansion wave and so test time is very short. Free-piston double-diaphragm shock tubes[4] produce shock velocities which exceed 17 km/sec. But when the test gas pressure is 1 or 0.5 torr, wave attenuation is very high; when initial test gas pressure is 0.1 torr, shock speed increases along the shock tube. In these circumstances the flow is not steady and radiative transfer problems are very severe. In our shock tube, wave attenuation is weak (200 m/sec per meter at 0.2 torr initial test gas pressure; 250 m/sec per meter at 1 torr initial test gas pressure) and a test-time of 10 μsec is available 10 m distant from D_2 diaphragm.

Use of an intermediate chamber and very high pressure in the driver increase combustion shock tube performance as high as that of powerful arc driven shock tubes.

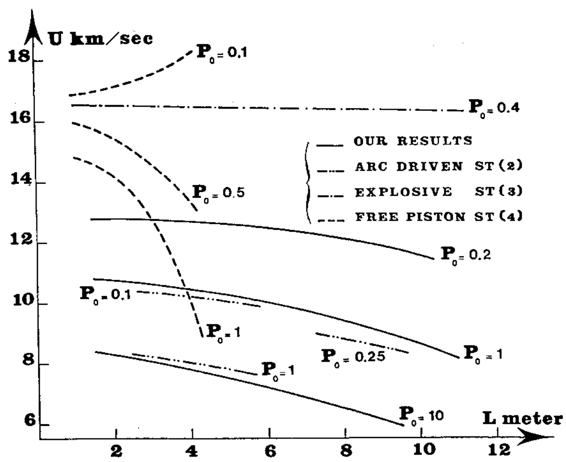

Fig. 3 Attenuation of shock wave velocity along the tube for some high performance shock tubes. P_0 is the initial test gas pressure in torr.

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Regularization of Grand Tour Trajectories

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Introduction

THE computation of precision Grand Tour interplanetary trajectories is very sensitive to errors introduced by the method of solution. To obtain accurate solutions of trajectories with multiple, consecutive, planetary close approaches, numerical integration of unregularized equations of motion often requires a large number of integration steps and excessive calculation times. To reduce the number of integration steps during each close approach, KS regularization, introduced by Kustaanheimo and Stiefel,[1] is employed. The regularization eliminates the singularities due to the planetary and solar attractions in the equations of motion of a space vehicle.

The simultaneous removal of all mathematical and numerical singularities in the equations of motion of a space vehicle, with both independent and dependent variable transformations, has not yet been accomplished. The KS method removes only one of the singularities. Since the space vehicle encounters the singularities only consecutively and not simultaneously, it appears sufficient to remove the singularities consecutively, one at a time, as the singularities are approached. A regularizing algorithm developed by Szebehely and Peters[2] is employed for this purpose. The algorithm applies the Levi-Civita method of regularization[3] to remove the singularity between the closest

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pair of the three bodies in the general, planar problem of three bodies. The singularities are removed consecutively, as they occur, and, excepting the case of a simultaneous triple collision, no singularities are encountered by any of the three bodies throughout their motion. The regularizing algorithm was generalized by Peters[4] for an arbitrary number of bodies in three-dimensions by using KS regularization. With some modification, the generalized, three-dimensional algorithm is extended here to interplanetary trajectories with consecutive, planetary close approaches. By eliminating the singularities, the truncation error is decreased at each step, and the total number of integration steps is reduced significantly, with a corresponding increase in accuracy and decrease in calculation time, compared with unregularized methods of solution.

The Regularizing Algorithm

The KS method of regularization is thoroughly discussed in the literature (for example, Stiefel and Scheifele[5]). Let $\mathbf{u}$ be the vector to the space vehicle in the KS four-dimensional space, in a coordinate system centered at the sun or at one of the planets. Let dots denote differentiation with respect to the real time t and primes denote differentiation with respect to the regularizing time. If $\mathbf{r}_i$ is the heliocentric position-vector of the ith planet, the equations of motion for the space vehicle and for the planets are

$$\mathbf{u}'' = \mathbf{\Phi}(\mathbf{u}, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p) \tag{1a}$$

$$t' = |\mathbf{u}|^2 \tag{1b}$$

$$\ddot{\mathbf{r}}_i \mathbf{\Psi}_i(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p) \tag{1c}$$

where $i = 1, 2, \ldots, p$, for p planets. Equation (1c) may be solved independently of Eqs. (1a) and (1b), producing planetary ephemerides tabulated in real time. In order to solve Eq. (1a) by discrete numerical methods, the planetary vectors must be known at the tabulations of the regularizing time in order to evaluate the function $\mathbf{\Phi}$. The relation between the real and the regularizing time is found through Eq. (1b) and interpolation of the planetary ephemerides may then be carried out. This approach has the advantage that Eq. (1c) needs to be solved only once. The disadvantages are that the planetary ephemerides must be stored and continually retrieved. Also, errors may result due to the interpolation process and due to the solution of Eq. (1b). The solution of Eq. (1b) often has errors of a greater magnitude than the solution of Eq. (1a). The solution for the vector $\mathbf{u}$ is stable with respect to the regularizing time for unperturbed motion since it is harmonic motion. But $\mathbf{u}$ is unstable with respect to the real time, in the Lyapunov sense, corresponding to "in-track" error of two body motion (for example, Szebehely[6] and Stiefel[7]). Hence there is an instability inherent in the solution of Eq. (1b), which, through the vectors $\mathbf{r}_i$, will negate, to some extent, the stability of the perturbed Eq. (1a).

An approach that removes most of these disadvantages is to simultaneously solve the following system of first-order differential equations

$$\mathbf{u}' = \mathbf{V}; \quad \mathbf{V}' = \mathbf{\Phi}(\mathbf{u}, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p) \tag{2a}$$

$$t' = |\mathbf{u}|^2 \tag{2b}$$

$$\mathbf{r}_i' = |\mathbf{u}|^2 \cdot \dot{\mathbf{r}}_i; \quad \dot{\mathbf{r}}_i' = |\mathbf{u}|^2 \cdot \mathbf{\Psi}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p) \tag{2c}$$

for $i = 1, 2, .., p$. The accuracy of the solution of the system of Eqs. (2) will generally be more accurate than the solution of Eqs. (1). The disadvantage of solving Eqs. (2) simultaneously is that Eqs. (2c) must be solved along with Eqs. (2a) and (2b) for each interplanetary trajectory. Also, the numerical integration of Eqs. (2c) proceeds with the same step-size as the integration of Eqs. (2a) and (2b), unless a technique is employed in which different step-sizes are allowed for each of the differential equations. The integration of Eqs. (2a) and (2b), for interplanetary trajectories with close approaches, often requires a smaller step-size than the integration of Eqs. (2c) for the planets. Hence, the planets are not integrated as efficiently as when Eq. (1c) is integrated separately.

The space vehicle is regularized with respect to a planet when the vehicle is within a fictitious sphere centered on the planet. When outside of all such planetary spheres, the vehicle is regularized with respect to sun. Two criteria to determine the radii of the spheres have been investigated. One is the classical sphere of influence defined by

$$\rho_i = |\mathbf{r}_i| \cdot m_i^{2/5} \tag{3}$$

where ρ_i is the radius of the sphere centered on the ith planet, $\mathbf{r}_i$ is the heliocentric position vector of the ith planet and m_i is the mass of the ith planet, in units of the solar mass. The classical spheres of influence are relatively small spheres about each planet. If regularization of the equations of the space vehicle with respect to a planet is made only when the vehicle is within the classical sphere, the vehicle would be regularized about the sun most of the time.

For trajectories with consecutive, planetary close approaches, as the vehicle leaves one planet and simultaneously approaches the next, the rates of change of the forces due to the departure planet and to the arrival planet are often larger than the rate of change of the solar force, even though the solar force is greater in magnitude for longer duration. Most methods of numerical integration with variable step-size determine the step-size to maintain a desired tolerance or bound on the local or per step truncation error. The truncation error is generally dominated by the force with the largest rate of change. Since regularization linearizes the effect of the singular forces, thereby decreasing the rates of change of these forces, it appears that radii of spheres of regularization should be determined by criteria reflecting not magnitudes of forces or closest distances, but relative rates of change or magnitudes of successive derivatives of each force. Optimally, for an nth-order method of numerical integration, it appears that the radii should be chosen so as to minimize the $(n+1)$ derivative [or $(n+1)$ difference] of the total force with respect to the independent variable, because this quantity is usually proportional to the local truncation error in most methods of numerical integration.

A second criterion was used to regularize the forces with nearly the largest rates of change and to enable the space vehicle to be regularized always with respect to a planet and never to the sun. The criterion provides enlarged spheres of influence centered on the planets by writing Eq. (3), which defines the classical spheres of influence, as

$$\rho_i/\rho_j = (m_i/m_j)^{2/5} \tag{4}$$

where ρ_i is the radius of the sphere centered on the ith planet and ρ_j is the sphere centered on the jth planet. Equation (4) is applied separately to each pair of planets as if only those two planets existed, neglecting the sun and the other planets. For each pair of planets there is a unique pair of spheres and the spheres are much larger than the classical spheres of influence. As the space vehicle departs the ith planet, about which it is regularized, and approaches the jth planet, the ratio $|\mathbf{r}-\mathbf{r}_i|/|\mathbf{r}-\mathbf{r}_j|$, where $\mathbf{r}$ is the heliocentric position vector of the space vehicle, increases until it becomes greater than the ratio ρ_i/ρ_j defined by Eq. (4). When this occurs, the regularized equations of motion are transformed from the ith planet to the jth planet.

For an Earth-Jupiter-Saturn interplanetary trajectory, with a 1977 launch date, the Earth-Jupiter pair has a sphere of radius 0.4 a.u. centered on the Earth and a sphere of radius 4.3 a.u. centered on Jupiter. When the space vehicle is at a distance of 0.4 a.u. from the Earth, the ratio is about equal to that given by Eq. (4) and the two spheres are in contact at one point. The Jupiter-Saturn pair has a sphere of radius 2.6 a.u. centered on Jupiter and a sphere of 2.0 a.u. centered on Saturn. When the vehicle is 2.6 a.u. from Jupiter, the two spheres are in contact.

When the space vehicle leaves a planetary sphere, the equations of motion are transformed to another planet or to the sun. For the Earth-Jupiter-Saturn trajectory, regularizing with respect to the planets when the vehicle is within the classical planetary spheres of influence and with respect to the sun when outside the planetary spheres requires four transformations, whereas use of the enlarged spheres requires only two transformations, one between Earth and Jupiter and the other between Jupiter and Saturn. Some numerical evidence has been found that significant digits may be lost in the transformations and hence it seems desirable to have as few transformations as possible, an additional advantage of the enlarged planetary spheres.

Numerical Comparison

In order to determine the merits of using the regularized equations of motion, it is necessary to determine whether or not the regularized solutions are obtained with a greater accuracy while using equal or less calculation time than unregularized solutions. A paper by E. Stiefel[7] is directed to this question. Using a discretization formula due to Henrici,[8] Stiefel shows that the solution by numerical integration of a circular orbit with unregularized equations of motion has a secular truncation error growth of approximately $(66/2880)\cdot(t/n)^4\cdot t$, for a fourth-order method, where t is the time after n integration steps. For regularized equations of motion, the corresponding secular truncation error is approximately $(1/1280)\cdot(t/n)^4\cdot t$. Since the ratio of the first error to the second error is 88 to 3, Stiefel points out that the numerical integration of the regularized equations of motion for circular motion is about 30 times more accurate than the integration of the unregularized equations. He then performs numerical experiments to determine the relative accuracy for eccentric orbits. For eccentricity equal to 0.5, the regularized solutions are 169 times more accurate (have less local truncation error) than unregularized solutions and for eccentricities 0.9, they are 12,160 times more accurate.

Since the regularized equations of motion provide smaller local truncation error than unregularized equations, a method of numerical integration, with automatic step-size control that bounds the local truncation, will take larger integration steps for the regularized equations than for the unregularized equations. For a particular solution from an initial time to a final time, the regularized and unregularized solutions will have the same local or per step truncation error, but the regularized solution will take fewer steps. The total or global truncation error is proportional to the number of steps and hence should be smaller for the regularized solution. In addition, the fewer number of steps required for the regularized solution allows a decrease in calculation time compared with the unregularized solution.

Another study was performed by Szebehely and Pierce,[9] applying the Birkhoff regularization[10] to Earth-to-moon trajectories in a planar, circular Earth-moon restricted problem of three bodies. They performed extensive numerical integrations of trajectories departing near the Earth and arriving near the moon. The solutions were obtained for the Birkhoff regularized equations of motion and for a set of unregularized equations. They concluded that for the regularized solutions, compared with the unregularized solutions: 1) the number of integration steps is reduced by a factor of three; 2) the number of integration step-size changes is greatly reduced; 3) the relative accuracy of the integration is significantly increased; and, 4) the machine computation time is reduced by at least 50%.

The regularizing algorithm discussed in the previous section is applied to an Earth-Jupiter-Saturn interplanetary trajectory to determine the relative accuracies and calculation times between regularized and unregularized solutions. The interplanetary trajectory leaves the Earth on Sept. 3, 1977, passes by Jupiter 665 days later at a distance of $7\cdot10^5$ km and passes by Saturn 1426 days later at a distance of $1.5\cdot10^7$ kms. The calculation of the trajectory was terminated at 1500 days. Two independent methods of numerical integration are applied to both the KS regularized and unregularized equations of motion so as to lessen the possibility of the results being dependent upon the method of integration. One method of integration is a tenth-order, finite-difference method with an Adams-Bashforth predictor and an Adams-Moulton corrector. The method allows a varying step-size using a Lagrangian method to restart the integration. The other method of numerical integration is an eighth-order, Runge-Kutta-Fehlberg method,[11] with variable step-size capability. Both methods of integration are applied to the set of regularized Eqs. (2), using both the classical and the enlarged spheres of influence. Both methods of integration are also applied to a set of unregularized equations of motion using heliocentric or planetocentric coordinate systems according to the same criteria used for determining the coordinate system for the regularized equations. For the particular trajectory used here regularization utilizing the enlarged spheres of influence provides somewhat more accurate solutions than regularization using the smaller classical spheres of influence. Two possible reasons for the improved accuracy using the enlarged spheres are discussed in the previous section.

The two integration methods, applied to both the regularized and the unregularized equations, provide four separate numerical solutions for the Earth-Jupiter-Saturn trajectory, each beginning with the same initial conditions at Earth. The four solutions are obtained for a wide range of accuracies by using many different truncation error tolerances. As the truncation error is decreased, with a corresponding increase in the total number of integration steps, all four solutions appear to converge to the same trajectory, which provides some confidence that the limiting solution is nearly the true solution. The magnitude of the difference in the position-vector of the space vehicle at the final time between a calculated solution and the limiting solution is taken as an estimate of the error of the calculated solution.

Figure 1 shows the error in the final position-vector as a function of the time of calculation for the trajectory. The calculations were performed on a CDC 6600 computer. The position error is given in km and the time of calculation in minutes. Curve A, in Fig. 1, represents solutions with different tolerances using the tenth-order, finite-difference method of numerical integration and unregularized equations of motion. Curve B represents solutions using the same finite-difference method but solving the regularized equations. Curve C represents solutions using the eighth-order, Runge-Kutta-Fehlberg method of integration solving the unregularized equations. And Curve D represents solutions using the same Runge-Kutta-Fehlberg method but solving the regularized equations. The data used to compose the curves of Fig. 1 have some dispersion about each curve of approximately three or four times the width of each curve. Also, the curves may not be continued to lower or higher accuracies by a simple extension of the curves. For lower

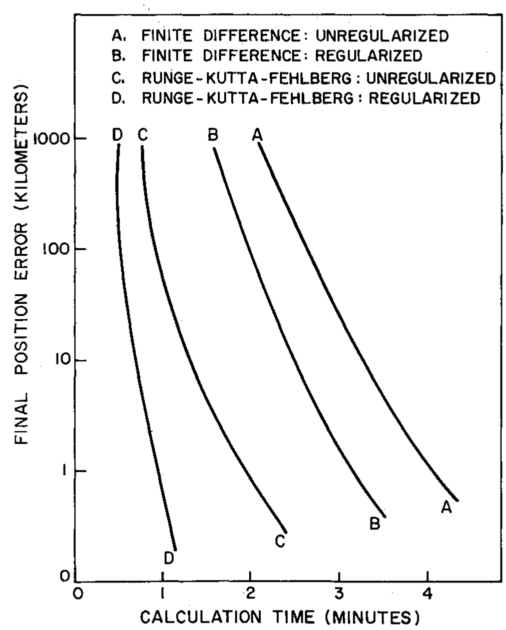

Fig. 1 Error in the final position vs time of calculation for the regularized and unregularized solutions.

accuracies, the solutions often become quite different because of sensitivity of the solutions due to the closeness of the planetary attraction. For higher accuracies, the curves quickly become nearly flat with little improvement in accuracy for large amounts of calculation time when using 14 digits of precision in the computations.

Comparing Curves A and B, for the finite-difference method of integration, the regularized equations of motion yield solutions with about 25% less calculation time than is required for the unregularized equations to attain the same accuracy. For the same calculation time, there is an increase in accuracy for the regularized solutions of about 10 to 20 times the unregularized solutions. Comparing Curves C and D, for the Runge-Kutta-Fehlberg method of integration, the regularized equations of motion yield solutions with about 35% to 45% less calculation time than is required for the unregularized equations to attain the same accuracy. For the same calculation time, there is an increase in accuracy for the regularized solutions of about 25 times the unregularized solutions.

A comparison between the finite-difference and the Runge-Kutta-Fehlberg methods of numerical integration for interplanetary trajectories is not intended here. The finite-difference method spends a significant amount of time in the Lagrangian starter for each step-size change and this time has been included in the calculation times of Fig. 1.

Conclusion

It appears that use of the regularized equations of motion yields solutions with greater accuracy while using less calculation time than unregularized equations, at least for the two methods of integration and for the particular trajectory used in the present study. The increased accuracy and less calculation time is due to the reduction in global truncation error and to the reduction in the number of integration steps. Due to the analytical result of Stiefel,[7] and the numerical results presented here and elsewhere,[9,12,13] it seems possible that the merits of the regularized equations of motion, as indicated in Fig. 1, may be extended to many other methods of integration and to the entire class of interplanetary trajectories with planetary close approaches.

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Fast Solution to Supersonic Plane Flat-Faced Blunt Body

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Nomenclature

c^* = sonic velocity
s, n = coordinate system
v_s, v_n = velocity components
v_{so} = body surface velocity (v_s at $n = 0$)
α = body incidence
δ = shock standoff distance, in n-direction for given s
χ = shock wave inclination relative to normal to freestream
γ = ratio of specific heats

THE first order, or one strip solution obtained by the method of integral relations,[1–3] has been found useful for accurately predicting inviscid high-speed flow about blunt bodies of arbitrary shape.[4–10] It has been found to give results indistinguishable from higher order solutions, and experiment, if freestream Mach number exceeds 8 (Ref. 1) and yields surprisingly good results in the neighborhood of the stagnation point at much lower Mach numbers.[3]

If, following Traugott,[6] we adopt body-oriented coordinates (s, n) using the stagnation point as origin (Fig. 1), the one strip method of integral relations reduces the transonic inviscid perfect gas flowfield equations to 3 simultaneous ordinary differential equations in the independent variable s for 3 dependent variables δ, χ, and v_{so}.

In the general case of a symmetric body of arbitrary shape, the problem can only be closed by applying a regularity condition through the sonic region singularity and iterating on the unknown

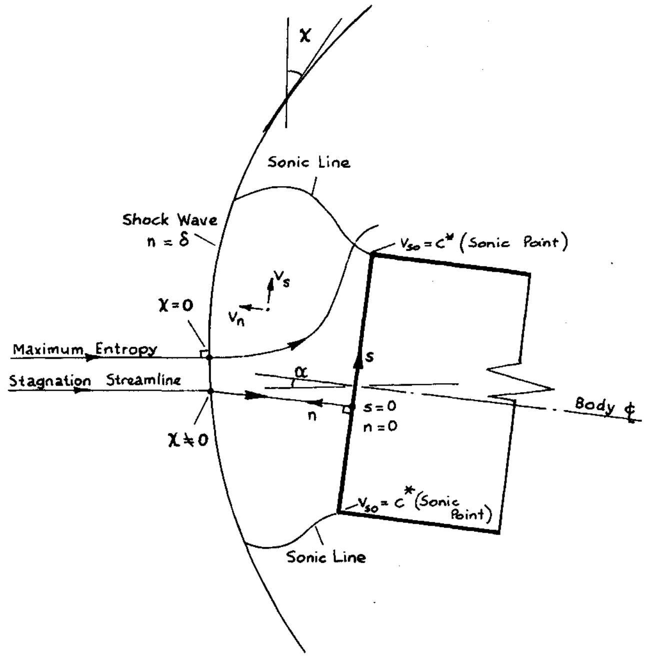

Fig. 1 Flowfield boundary and coordinate system.

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